Math Meth Oper Res (2005) 61:329–343
DOI 10.1007/s001860400385

Mathematical Methods
of Operations Research

Cost benefit analysis of series systems with warm standby components and general repair time

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Manuscript received: October 2003/Final version received: July 2004

Abstract. We study the availability analysis of three different series system configurations with warm standby components and general repair times. The time-to-failure for each of the primary and warm standby components is assumed to be exponentially distributed with respective parameter λ and α. This paper presents a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the steady-state probability distribution of the number of working components in the system. We develop the explicit expressions for the steady-state availability, Av, for three configurations and perform comparisons. For all three configurations, comparisons are made for specific values of distribution parameters and of the cost of the components. The configurations are ranked based on Av and cost/benefit, for three various repair time distributions: exponential, 3-stage Erlang, and deterministic, where benefit is Av.

Key words: Availability, Cost-benefit, Recursive method, Series system, Supplementary variable, General repair times

1 Introduction

In this paper we use a supplementary variable technique to study the availability analysis of three different series system configurations with warm standby components. The steady-state availability, Av, has widely been analyzed in the literature because of its prevalence in power plants, manufacturing systems, and industrial systems. Maintaining a high or required level of availability is often an essential requisite. A standby component is called a 'warm standby' if its failure rate is nonzero and is less than the failure rate of a primary component. Primary and warm standby components can be considered to be repairable.

The supplementary variable technique was first proposed by Cox [2], and it had been widely applied to the M/G/1 queueing system by Cohen [1], Hokstad [6], Keilson and Kooharian [7], Takacs [8] and many others. On the basis of this technique, Gupta and Rao [4, 5] studied the no-spare M/G/1 machine repair problem and the cold-standby M/G/1 machine repair problem, respectively. Gaikowsky, et al. [3] and Wang and Pearn [9] analyzed the series systems with cold standby components and warm standby components, respectively, where the repair time distribution of the server is assumed to be exponentially distributed. Wang and Kuo [10] dealed the reliability, the availability, and the cost/benefit analysis of four different series system configurations with mixed standby (include cold standby and warm standby) components.

The problem considered in this paper is more general than the works of Gaikowsky, et al. [3] and Wang and Pearn [9]. The explicit expression for the Av to series system with repair time distribution of the general type has not been found. We first present a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the Av_i, for configuration i, where $i = 1, 2, 3$. Next, for each configuration, the explicit expressions for the Av for three different repair time distributions such as exponential (M), k-stage Erlang (E_k), and deterministic (D) are provided. Finally, we rank three configurations for the Av based on assumed numerical values given to the system parameters.

2. Description of the system

For the sake of discussion, we consider the requirements of a 30 MW power plant. We assume that generators are available in units of 30 MW, 15 MW and 10 MW. We also assume that standby generators are allowed to fail while inactive before they are put into full operation, and that the standby generators are continuously monitored by a fault detecting device in order to identify if they fail or not. It is assumed that all switchover times are instantaneous and switching is perfect, e.g. never fails and never does any damage. Primary components and warm standby components can be considered to be repairable. Suppose that each of the primary components fails independently of the state of the others and has an exponential time-to-failure distribution with parameter λ. Whenever one of these components fails, it is immediately replaced by a warm standby component if any is available. We now assume that each of the available standby components fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter α $(0 < \lambda < \alpha)$. It is assumed that the time-to-repair of the components are independent and identically distributed (i.i.d.) random variables having a distribution $B(u)(u \geq 0)$, a probability density function $b(u)(u \geq 0)$ and mean service time b_1. If one component is in repair, then arriving failed components have to wait in the queue until the server is available. Let us assume that failed components arriving at the server form a single waiting line and are served in the order of their arrivals; i.e., according to the first-come, first-served discipline. Suppose that the server can serve only one primary component (or warm standby component) at a time, and that the service is independent of the arrival of the components. Once a component is repaired, it is as good as new.

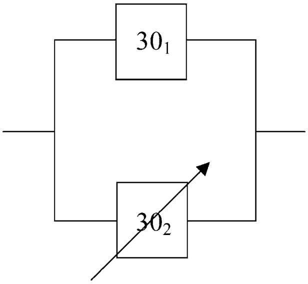

Fig. 1. Configuration 1□ one primary 30MW component and one warm standby 30MW component

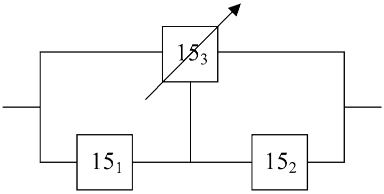

Fig. 2. Configuration 2□ two primary 15 MW components and one warm standby 15 MW component

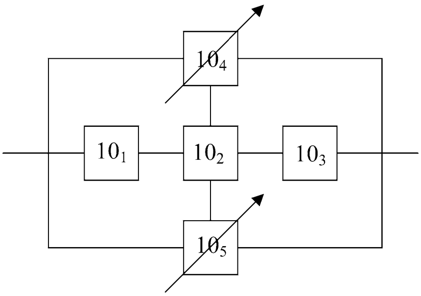

Fig. 3. Configuration 3□ three primary 10 MW components and two interchangeable 10 MW warm standby component

We consider three configurations as follows: The first configuration is a serial system of one primary 30 MW component with one warm standby 30 MW component (Figure 1). The second configuration is a serial system of two primary 15 MW components and one warm standby 15 MW component (Figure 2). The standby unit can replace either one of the initially working units in case of failure. The last configuration is a serial system of three primary 10 MW components with two interchangeable warm standby 10 MW components. (Figure 3). Each standby unit can replace either one of the failed components. If necessary, a standby unit can also replace the first used standby unit in case of failure.

2.1. Cost-benefit factor

We assume that the size-proportional costs for the primary components and warm standby components are given in Table 1. With this, we calculate the costs for each configuration $i (i = 1, 2, 3)$ shown in Table 2. Let C_i be the cost of the configuration i, and B_i be the benefit of the configuration i, where B_i is the Av_i.

Table 1. The size-proportional cost for the primary and warm standby components

Component	Cost (in $)
Primary 30 MW	30×10^6
Primary 15 MW	15×10^6
Primary 10 MW	10×10^6
Warm standby 30 MW	18×10^6
Warm standby 15 MW	9×10^6
Warm standby 10 MW	6×10^6

Table 2. The costs for each configuration i, $i = 1, 2, 3$

Configuration	Cost (in $)
Configuration 1	48×10^6
Configuration 2	39×10^6
Configuration 3	42×10^6

3. Availability analysis of the system

We use the following supplementary variable: $U \equiv$ remaining repair time for the component under repair. The state of the system at time t is given by

$N(t) \equiv$ number of working components in the system, and

$U(t) \equiv$ remaining repair time for the component being repaired.

Let us define

$$P_n(u,t)du = Pr\{N(t) = n, u < U(t) \le u + du\}, \quad u \ge 0$$

$$P_n(t) = \int_0^\infty P_n(u,t)du.$$

3.1. *Availability for configuration 1*

Relating the state of the system at time t and $t + dt$, we obtain

$$\frac{d}{dt}P_2(t) = -(\lambda + \alpha)P_2(t) + P_1(0,t), \tag{1}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_1(u,t) = -\lambda P_1(u,t) + (\lambda + \alpha)P_2(u,t) + P_0(0,t)b(u), \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_0(u,t) = \lambda P_1(u,t). \tag{3}$$

In steady-state, let us define

$$P_n = \lim_{t\to\infty} P_n(t), \quad n = 2, 1, 0$$
$$P_n(u) = \lim_{t\to\infty} P_n(u,t), \quad n = 2, 1, 0.$$

and further define

$$P_2(u) = P_2 b(u). \tag{4}$$

From (1)–(4), the steady-state equations are given by:

$$0 = -(\lambda + \alpha)P_2 + P_1(0), \tag{5}$$

$$-\frac{d}{du}P_1(u) = -\lambda P_1(u) + (\lambda + \alpha)P_2 b(u) + P_0(0)b(u), \tag{6}$$

$$-\frac{d}{du}P_0(u) = \lambda P_1(u). \tag{7}$$

It follows from (5)

$$P_1(0) = (\lambda + \alpha)P_2. \tag{8}$$

Further define

$$B^*(s) = \int_0^\infty e^{-su}dB(u) = \int_0^\infty e^{-su}b(u)du,$$

$$P_n^*(s) = \int_0^\infty e^{-su}P_n(u)du,$$

$$P_n = P_n^*(0) = \int_0^\infty P_n(u)du,$$

and

$$\int_0^\infty e^{-su}\frac{d}{du}P_n(u)du = sP_n^*(s) - P_n(0).$$

Taking the LST on both sides of (6)–(7) and using (8), we get

$$(\lambda - s)P_1^*(s) = P_1(0)B^*(s) + P_0(0)B^*(s) - P_1(0), \tag{9}$$

$$-sP_0^*(s) = \lambda P_1^*(s) - P_0(0). \tag{10}$$

A recursive method is used to develop the steady-state solutions $P_n^*(0)(n = 1, 0)$. Setting $s = \lambda$ and $s = 0$ in (9), respectively, we finally obtain

$$P_0(0) = \frac{1 - B^*(\lambda)}{B^*(\lambda)}P_1(0) = \frac{(\lambda + \alpha)[1 - B^*(\lambda)]}{B^*(\lambda)}P_2, \tag{11}$$

and

$$P_1^*(0) = \frac{1}{\lambda}P_0(0) = \frac{(\lambda + \alpha)\left[1 - B^*(\lambda)\right]}{\lambda B^*(\lambda)}P_2. \tag{12}$$

Differentiating (10) with respect to s and setting $s = 0$ finally gives

$$P_0^*(0) = -\lambda P_1^{*(1)}(0). \tag{13}$$

Likewise, differentiating (9) with respect to s and setting $s = 0$ finally yields

$$\lambda P_1^{*(1)}(0) = P_1^*(0) - b_1\left[P_0(0) + P_1(0)\right], \tag{14}$$

where $b_1 = -B^{*(1)}(0)$ denotes the mean repair time.

From (13)–(14), we have

$$P_0^*(0) = -P_1^*(0) + b_1\left[P_0(0) + P_1(0)\right]. \tag{15}$$

Using (8) and (11)–(12) on (15) finally yields

$$P_0^*(0) = \frac{(\lambda+\alpha)\left[\lambda b_1 - 1 + B^*(\lambda)\right]}{\lambda B^*(\lambda)} P_2. \tag{16}$$

In order to find P_2, we substitute (12) and (16) in the normalizing condition

$$P_2 + P_1^*(0) + P_0^*(0) = 1,$$

which gives

$$P_2 = \frac{B^*(\lambda)}{b_1(\lambda+\alpha) + B^*(\lambda)}. \tag{17}$$

Thus

$$P_0^*(0) = \frac{(\lambda+\alpha)\left[\lambda b_1 - 1 + B^*(\lambda)\right]}{\lambda\left[b_1(\lambda+\alpha) + B^*(\lambda)\right]}. \tag{18}$$

For configuration 1, the explicit expression for the Av_1 is given by

$$Av_1 = 1 - P_0^*(0) = \frac{\lambda + \alpha\left[1 - B^*(\lambda)\right]}{\lambda\left[b_1(\lambda+\alpha) + B^*(\lambda)\right]}. \tag{19}$$

3.1.1. Special cases

We present three special cases for three different repair time distributions such as exponential (M), k-stage Erlang (E_k), and deterministic (D). The explicit expressions for the Av_1 for three different repair time distributions such as exponential, k-stage Erlang, and deterministic are given in the following.

Case 1. The repair time has exponential distribution. We set the mean repair time $b_1 = 1/\mu$, where μ is the repair rate. In this case, we have

$$B^*(s) = \frac{\mu}{\mu+s}.$$

From (19), the explicit expression for the $Av_1(\mathrm{M})$ is given by

$$Av_1(\mathrm{M}) = \frac{\mu(\lambda+\mu+\alpha)}{\lambda(\lambda+\alpha) + \mu(\lambda+\mu+\alpha)}. \tag{20}$$

Case 2. The repair time has k-stage Erlang distribution. The k-stage Erlang distribution is made up of k independent and identical exponential stages, each with mean $1/k\mu$. We set the mean repair time $b_1 = 1/\mu$. In this case, we have

$$B^*(s) = \left(\frac{k\mu}{k\mu+s}\right)^k.$$

From (19) again, we obtain the following explicit expression for the $Av_1(\mathrm{E}_k)$

$$Av_1(\mathrm{E_k}) = \frac{\mu(\lambda+\alpha)(\lambda+k\mu)^k - \mu\alpha(k\mu)^k}{\lambda(\lambda+\alpha)(\lambda+k\mu)^k + \lambda\mu(k\mu)^k}. \tag{21}$$

Case 3. The repair time distribution is deterministic. We set the mean repair time $b_1 = 1/\mu$. In this case, we have

$$B^*(s) = e^{-s/\mu}.$$

It follows from (19) that the explicit expression for the $Av_1(\mathrm{D})$ is given by

$$Av_1(\mathrm{D}) = \frac{\mu\left(\lambda + \alpha - \alpha e^{-\frac{\lambda}{\mu}}\right)}{\lambda\left(\lambda + \alpha + \mu e^{-\frac{\lambda}{\mu}}\right)}. \tag{22}$$

3.2. Availability for configuration 2

Following the same procedures given in the section that analyzes the configuration 1 case, we can easily set up the steady-state equations as follows:

$$0 = -(2\lambda + \alpha)P_3 + P_2(0), \tag{23}$$

$$-\frac{d}{du}P_2(u) = -2\lambda P_2(u) + (2\lambda + \alpha)P_3 b(u) + P_1(0)b(u), \tag{24}$$

$$-\frac{d}{du}P_1(u) = 2\lambda P_2(u), \tag{25}$$

where we define

$$P_3(u) = P_3 b(u). \tag{26}$$

From (23) we have

$$P_2(0) = (2\lambda + \alpha)P_3. \tag{27}$$

Taking the LST on both sides of (24)–(25) and using (27), we get

$$(2\lambda - s)P_2^*(s) = P_2(0)B^*(s) + P_1(0)B^*(s) - P_2(0), \tag{28}$$

$$-sP_1^*(s) = 2\lambda P_2^*(s) - P_1(0). \tag{29}$$

Setting $s = 2\lambda$ and $s = 0$ in (28), respectively, we finally get

$$P_1(0) = \frac{1 - B^*(2\lambda)}{B^*(2\lambda)}P_2(0) = \frac{(2\lambda + \alpha)\left[1 - B^*(2\lambda)\right]}{B^*(2\lambda)}P_3, \tag{30}$$

and

$$P_2^*(0) = \frac{1}{2\lambda}P_1(0) = \frac{(2\lambda + \alpha)\left[1 - B^*(2\lambda)\right]}{2\lambda B^*(2\lambda)}P_3. \tag{31}$$

Differentiating (29) with respect to s and setting $s = 0$ finally get

$$P_1^*(0) = -2\lambda P_2^{*(1)}(0). \tag{32}$$

Likewise, differentiating (28) with respect to s and setting $s = 0$ finally obtain

$$2\lambda P_2^{*(1)}(0) = P_2^*(0) - b_1\left[P_1(0) + P_2(0)\right]. \tag{33}$$

It implies from (32)–(33) that

$$P_1^*(0) = -P_2^*(0) + b_1\left[P_1(0) + P_2(0)\right]. \tag{34}$$

Using (27) and (30)–(31) finally gives

$$P_1^*(0) = \frac{(2\lambda + \alpha)\left[2\lambda b_1 - 1 + B^*(2\lambda)\right]}{2\lambda B^*(2\lambda)} P_3. \tag{35}$$

In order to find P_3, we substitute (31) and (35) in the normalizing condition

$P_3 + P_2^*(0) + P_1^*(0) = 1,$

which yields

$$P_3 = \frac{B^*(2\lambda)}{b_1(2\lambda + \alpha) + B^*(2\lambda)}. \tag{36}$$

Thus

$$P_1^*(0) = \frac{(2\lambda + \alpha)\left[2\lambda b_1 - 1 + B^*(2\lambda)\right]}{2\lambda\left[b_1(2\lambda + \alpha) + B^*(2\lambda)\right]}. \tag{37}$$

For configuration 2, the explicit expression for the Av_2 is given by

$$Av_2 = 1 - P_1^*(0) = \frac{2\lambda + \alpha\left[1 - B^*(2\lambda)\right]}{2\lambda\left[b_1(2\lambda + \alpha) + B^*(2\lambda)\right]}. \tag{38}$$

3.2.1. Special cases

For configuration 2, we also consider three special cases for three different repair time distributions such as exponential (M), k-stage Erlang (E_k), and deterministic (D). We provide the following explicit expressions for the $Av_2(\mathrm{M})$, $Av_2(\mathrm{E_k})$, $Av_2(\mathrm{D})$ for three different repair time distributions: exponential, k-stage Erlang, and deterministic, respectively.

$$Av_2(\mathrm{M}) = \frac{\mu(2\lambda + \mu + \alpha)}{2\lambda(2\lambda + \alpha) + \mu(2\lambda + \mu + \alpha)}. \tag{39}$$

$$Av_2(\mathrm{E_k}) = \frac{\mu(2\lambda + \alpha)(2\lambda + k\mu)^k - \mu\alpha(k\mu)^k}{2\lambda(2\lambda + \alpha)(2\lambda + k\mu)^k + 2\lambda\mu(k\mu)^k}. \tag{40}$$

$$Av_2(\mathrm{D}) = \frac{\mu\left(2\lambda + \alpha - \alpha e^{-\frac{2\lambda}{\mu}}\right)}{2\lambda\left(2\lambda + \alpha + \mu e^{-\frac{2\lambda}{\mu}}\right)}. \tag{41}$$

3.3. Availability for configuration 3

Following the same procedures given in the section that analyzes the configurations 1 and 2 cases, we can easily set up the steady-state equations as follows:

$$0 = -(3\lambda + 2\alpha)P_5 + P_4(0), \tag{42}$$

$$-\frac{d}{du}P_4(u) = -(3\lambda + \alpha)P_4(u) + (3\lambda + 2\alpha)P_5 b(u) + P_3(0)b(u), \tag{43}$$

$$-\frac{d}{du}P_3(u) = -3\lambda P_3(u) + (3\lambda + \alpha)P_4(u) + P_2(0)b(u), \tag{44}$$

$$-\frac{d}{du}P_2(u) = 3\lambda P_3(u), \tag{45}$$

where we define

$$P_5(u) = P_5 b(u). \tag{46}$$

From (42), we have

$$P_4(0) = (3\lambda + 2\alpha)P_5. \tag{47}$$

Taking the LST on both sides of (43)–(45) and using (47), it yields

$$(3\lambda + \alpha - s)P_4^*(s) = P_4(0)B^*(s) + P_3(0)B^*(s) - P_4(0), \tag{48}$$

$$(3\lambda - s)P_3^*(s) = (3\lambda + \alpha)P_4^*(s) + P_2(0)B^*(s) - P_3(0), \tag{49}$$

$$-sP_2^*(s) = 3\lambda P_3^*(s) - P_2(0). \tag{50}$$

Setting $s = 3\lambda + \alpha$ and $s = 0$ in (48), respectively, we finally obtain

$$P_3(0) = \frac{1 - B^*(3\lambda + \alpha)}{B^*(3\lambda + \alpha)} P_4(0) = \frac{(3\lambda + 2\alpha)\left[1 - B^*(3\lambda + \alpha)\right]}{B^*(3\lambda + \alpha)} P_5, \tag{51}$$

and

$$P_4^*(0) = \frac{1}{3\lambda + \alpha} P_3(0) = \frac{(3\lambda + 2\alpha)\left[1 - B^*(3\lambda + \alpha)\right]}{(3\lambda + \alpha)B^*(3\lambda + \alpha)} P_5. \tag{52}$$

Likewise, setting $s = 3\lambda$ and $s = 0$ in (49), respectively, we finally get

$$P_2(0) = \frac{P_3(0) - (3\lambda + \alpha)P_4^*(3\lambda)}{B^*(3\lambda)}, \tag{53}$$

and

$$P_3^*(0) = \frac{1}{3\lambda} P_2(0). \tag{54}$$

Differentiating (50) with respect to s and setting $s = 0$ finally yields

$$P_2^*(0) = -3\lambda P_3^{*(1)}(0). \tag{55}$$

Differentiating (49) with respect to s and setting $s = 0$, we finally get

$$3\lambda P_3^{*(1)}(0) = P_3^*(0) + (3\lambda + \alpha)P_4^{*(1)}(0) - b_1 P_2(0). \tag{56}$$

Likewise, differentiating (48) with respect to s and setting $s = 0$, we have

$$(3\lambda + \alpha)P_4^{*(1)}(0) = P_4^*(0) - b_1\left[P_3(0) + P_4(0)\right]. \tag{57}$$

From (55)–(57), we finally obtain

$$P_2^*(0) = -\left[P_3^*(0) + P_4^*(0)\right] + b_1\left[P_2(0) + P_3(0) + P_4(0)\right]. \tag{58}$$

Setting $s = 3\lambda$ in (48) yields

$$P_4^*(3\lambda) = \frac{(3\lambda + 2\alpha)\left[B^*(3\lambda) - B^*(3\lambda + \alpha)\right]}{\alpha B^*(3\lambda + \alpha)} P_5. \tag{59}$$

Substituting (51) and (59) into (53), we finally get

$$P_2(0) = \frac{(3\lambda + 2\alpha)\left\{\alpha\left[1 - B^*(3\lambda)\right] + 3\lambda\left[B^*(3\lambda + \alpha) - B^*(3\lambda)\right]\right\}}{\alpha B^*(3\lambda)B^*(3\lambda + \alpha)} P_5. \tag{60}$$

We must again use the following normalizing condition

$$P_5 + P_4^*(0) + P_3^*(0) + P_2^*(0) = 1,$$

in (58), which gives

$$P_5 + b_1\left[P_2(0) + P_3(0) + P_4(0)\right] = 1. \tag{61}$$

Utilizing (47), (51), and (60) in (61), we obtain

$$P_5 = \left[1 + \frac{b_1(3\lambda + 2\alpha)\left\{\alpha + 3\lambda\left[B^*(3\lambda + \alpha) - B^*(3\lambda)\right]\right\}}{\alpha B^*(3\lambda)B^*(3\lambda + \alpha)}\right]^{-1}. \tag{62}$$

For configuration 3, the explicit expression for the Av_3 is given by

$$\begin{aligned} Av_3 &= 1 - P_2^*(0) = P_5 + P_4^*(0) + P_3^*(0) \\ &= \left[1 + \frac{(3\lambda + 2\alpha)\left[1 - B^*(3\lambda + \alpha)\right]}{(3\lambda + \alpha)B^*(3\lambda + \alpha)}\right. \\ &\quad \left. + \frac{(3\lambda + 2\alpha)\left\{\alpha\left[1 - B^*(3\lambda)\right] + 3\lambda\left[B^*(3\lambda + \alpha) - B^*(3\lambda)\right]\right\}}{3\lambda\alpha B^*(3\lambda)B^*(3\lambda + \alpha)}\right]P_5, \end{aligned} \tag{63}$$

where P_5 is given in (62).

For configuration 3, substituting the expressions $B^*(s)$ for three various repair time distributions: exponential, k-stage Erlang, and deterministic into (63), we can obtain the explicit expressions for the $Av_3(\mathrm{M})$, $Av_3(\mathrm{E_k})$, $Av_3(\mathrm{D})$, respectively. The explicit expressions for the $Av_3(\mathrm{M})$, $Av_3(\mathrm{E_k})$ and $Av_3(\mathrm{D})$ are too ample to be shown here. However, a numerical example will be given for this case to compare this configuration with other configurations.

4. Comparison of the three configurations

4.1. Comparison of the A_v

In this section, the computer software, e.g., MATLAB, is used to compare three configurations in terms of their $Av_i (i = 1, 2, 3)$ for three different repair time distributions: exponential, 3-stage Erlang, and deterministic.

We first perform a comparison for the Av of the configurations 1, 2, and 3 when the repair time distribution is exponential, or 3-stage Erlang, or deterministic. We choose $\alpha = 0.05$, $\mu = 1.5$, and vary the values of λ from 0.2 to 1.6. Numerical results of the $Av_i(\mathrm{M})$, $Av_i(\mathrm{E_3})$, and $Av_i(\mathrm{D})$ for configuration $i(i = 1, 2, 3)$ are shown in Table 3 for cases 1–3, respectively.

Next, we perform a comparison for the Av of the configurations 1, 2, and 3 when the repair time distribution is exponential, or 3-stage Erlang, or deterministic. We choose $\alpha = 0.05$, $\lambda = 0.2$, and vary the values of μ from 0.5 to 3.0. Numerical results of the $Av_i(\mathrm{M})$, $Av_i(\mathrm{E_3})$, and $Av_i(\mathrm{D})$ for configuration $i(i = 1, 2, 3)$ are shown in Table 4 for cases 1–3, respectively.

Table 3. Comparison of the configurations 1, 2, 3 for A_v

Range of λ	Result
Case 1. Exponential	
$0.2 < \lambda < 0.2984$	$Av_1(\mathrm{M}) > Av_3(\mathrm{M}) > Av_2(\mathrm{M})$
$0.2984 < \lambda < 1.6$	$Av_1(\mathrm{M}) > Av_2(\mathrm{M}) > Av_3(\mathrm{M})$
Case 2. 3-stage Erlang	
$0.2 < \lambda < 0.3658$	$Av_1(\mathrm{E}_3) > Av_3(\mathrm{E}_3) > Av_2(\mathrm{E}_3)$
$0.3658 < \lambda < 1.6$	$Av_1(\mathrm{E}_3) > Av_2(\mathrm{E}_3) > Av_3(\mathrm{E}_3)$
Case 3. Deterministic	
$0.2 < \lambda < 0.4058$	$Av_1(\mathrm{D}) > Av_3(\mathrm{D}) > Av_2(\mathrm{D})$
$0.4058 < \lambda < 1.6$	$Av_1(\mathrm{D}) > Av_2(\mathrm{D}) > Av_3(\mathrm{D})$

Table 4. Comparison of the configurations 1, 2, 3 for A_v

Range of μ	Result
Case 1. Exponential	
$0.5 < \mu < 1.0362$	$Av_1(\mathrm{M}) > Av_2(\mathrm{M}) > Av_3(\mathrm{M})$
$1.0362 < \lambda < 3.0$	$Av_1(\mathrm{M}) > Av_3(\mathrm{M}) > Av_2(\mathrm{M})$
Case 2. 3-stage Erlang	
$0.5 < \mu < 0.8447$	$Av_1(\mathrm{E}_3) > Av_2(\mathrm{E}_3) > Av_3(\mathrm{E}_3)$
$0.8472 < \mu < 3.0$	$Av_1(\mathrm{E}_3) > Av_3(\mathrm{E}_3) > Av_2(\mathrm{E}_3)$
Case 3. Deterministic	
$0.5 < \mu < 0.7578$	$Av_1(\mathrm{D}) > Av_2(\mathrm{D}) > Av_3(\mathrm{D})$
$0.7578 < \mu < 3.0$	$Av_1(\mathrm{D}) > Av_3(\mathrm{D}) > Av_2(\mathrm{D})$

4.2. Comparison of three configurations based on their cost/benefit ratios

We consider that the various configurations may have different costs when comparing all configurations. From Table 2, the cost (C_i) of the configuration $i(i = 1, 2, 3)$ are listed in the following:

$$C_1 = \$48 \times 10^6, C_2 = \$39 \times 10^6, C_3 = \$42 \times 10^6.$$

We first fix $\alpha = 0.05, \mu = 1.0$, and vary the values of λ from 0.2 to 1.6. Numerical results of the $C_i/Av_i(\mathrm{M})$, $C_i/Av_i(\mathrm{E}_3)$, and $C_i/Av_i(\mathrm{D})$ for configuration $i(i = 1, 2, 3)$ are depicted in Figures 4–6, respectively. Figures 4–6 show that the C_i/Av_i increases as λ increases for any configuration. One observes from Figure 4 that the optimal configuration using the $C_i/Av_i(\mathrm{M})$ value depends on the value of λ. When $0.2 < \lambda < 0.6308$, the optimal configuration is configuration 2, but when $0.6308 < \lambda < 1.6$, the optimal configuration is configuration 1. We can easily see from Figure 5 that the optimal configuration using the $C_i/Av_i(\mathrm{E}_3)$ value depends on the value of λ. When $0.2 < \lambda < 0.6913$, the optimal configuration is configuration 2, but when $0.6913 < \lambda < 1.6$, the optimal configuration is configuration 1. It appears from Figure 6 that the optimal configuration using the $C_i/Av_i(\mathrm{D})$ value depends on the value of λ. When $0.2 < \lambda < 0.7398$, the optimal configuration is configuration 2, but when $0.7398 < \lambda < 1.6$, the optimal configuration is configuration 1.

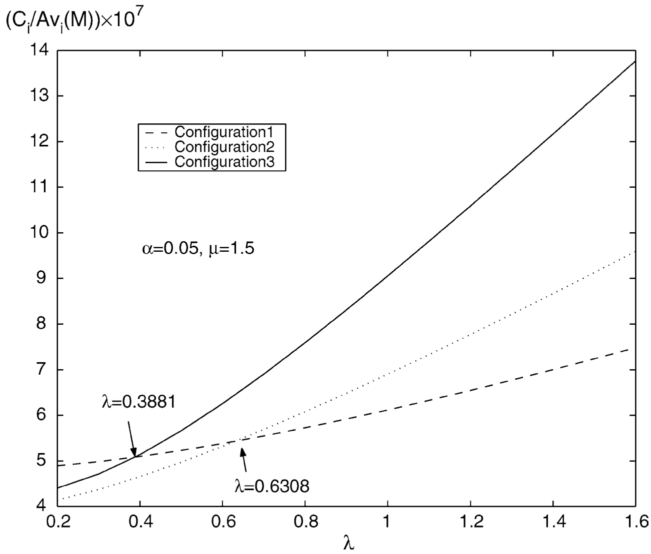

Fig. 4. $C_i/AV_i(M)$ versus failure rate λ

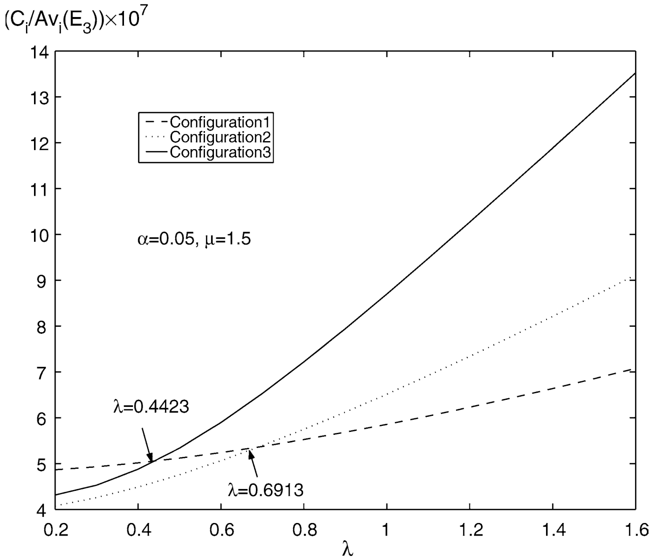

Fig. 5. $C_i/AV_i(E_3)$ versus failure rate λ

Next, we fix $\alpha = 0.05, \lambda = 0.2$, and vary the values of μ from 0.5 to 3.0. Numerical results of the $C_i/Av_i(\mathrm{M})$, $C_i/Av_i(\mathrm{E}_3)$, and $C_i/Av_i(\mathrm{D})$ for configuration $i(i = 1, 2, 3)$ are depicted in Figures 7–9, respectively. Figures 7–9 show that the C_i/Av_i decreases as μ increases for any configuration. We observe from Figures 7–9 that the optimal configuration using the $C_i/Av_i(\mathrm{M})$, or $C_i/Av_i(\mathrm{E}_3)$ or $C_i/Av_i(\mathrm{D})$ value is configuration 2.

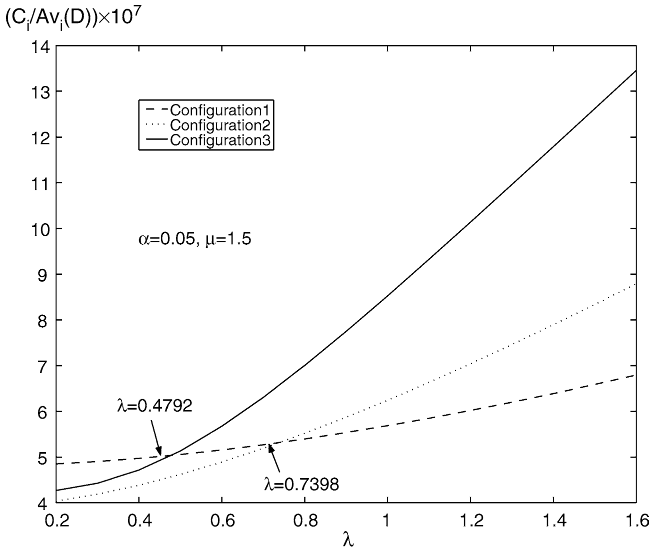

Fig. 6. $C_i/AV_i(D)$ versus failure rate λ

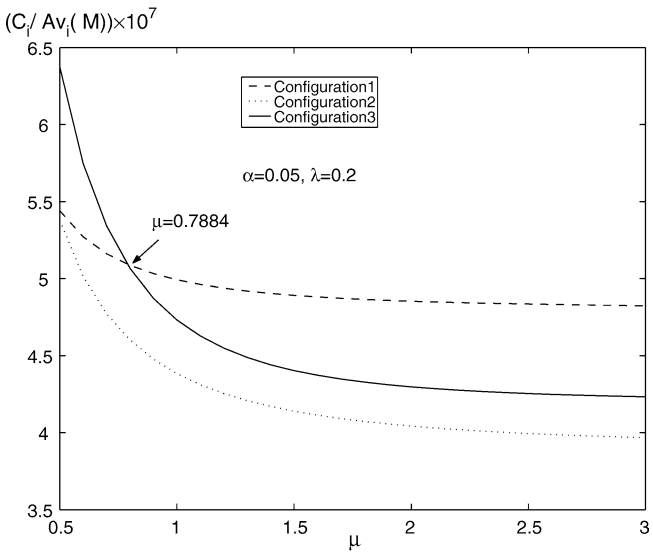

Fig. 7. $C_i/AV_i(M)$ versus service rate μ

5. Conclusions

In this paper we have first utilized the supplementary variable technique to develop the steady-state availability, Av, of three different series system configurations with warm standby components and general repair times.

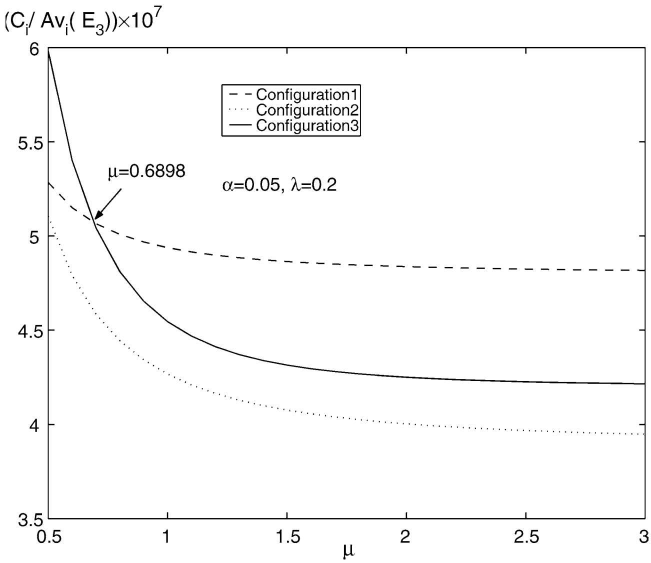

Fig. 8. C_i/AV_i (E_3) versus service rate μ

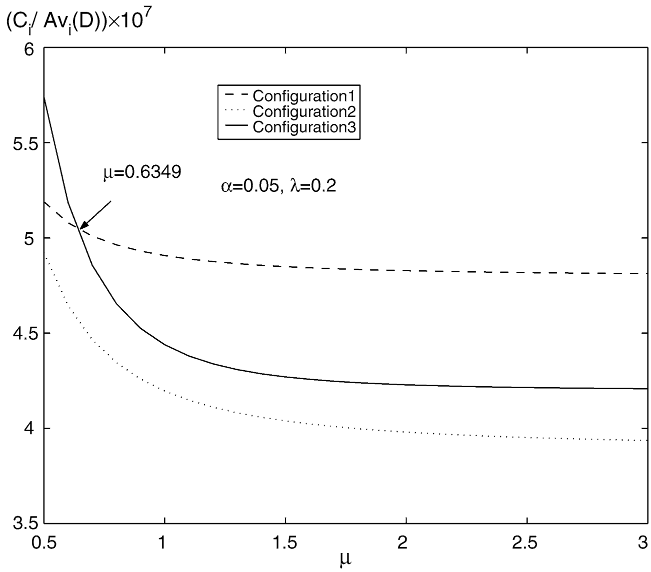

Fig. 9. C_i/AV_i (D) versus service rate μ

Next, for each configuration, we present the explicit expressions for the Av for three various repair time distributions such as exponential (M), k-stage Erlang (E_k), and deterministic (D). Finally, we rank three configurations based on the Av and the cost/Av for three various repair time distributions.

References

1. Cohen JW (1969) The single server queue. North-Holland, Amsterdam
2. Cox DR (1955) The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables. Proceedings of the Cambridge Philosophical Society 51:433–441.
3. Galikowsky C, Sivazlian BD, Chaovalitwongse P (1996) Optimal redundancies for reliability and availability of series systems. Microelectronics and Reliability 36:1537–1546
4. Gupta UC, Rao S (1994) A recursive method to compute the steady state probabilities of the machine interference model:(M/G/1)/K. Computers and Operations Research 21:597–605
5. Gupta UC, Rao S (1996) On the M/G/1 machine interference model with spares. European Journal of Operational Research 89:164–171
6. Hokstad P (1975) A supplementary variable technique applied to the M/G/1 queue. Scandinavian Journal of Statistics 2:95–98
7. Keilson J, Kooharian A (1960) On time dependent queueing processes. Annals of Mathmetical Statistics 31:104–112
8. Takacs L (1963) Delay distributions for one line with Poisson input, general holding times and various orders of service. Bell System Technique Journal 42:487–504
9. Wang KH, Pearn WL (2003) Cost benefit analysis of series systems with warm standby components. Mathematical Methods of Operations Research 58:247–258
10. Wang KH, Kuo CC (2000) Cost and probabilistic analysis of series systems with mixed standby components. Applied Mathematical Modelling 24:957–967